

ON POSITIVITY IN T -EQUIVARIANT K -THEORY OF FLAG VARIETIES

WILLIAM GRAHAM AND SHRAWAN KUMAR

ABSTRACT. We prove some general results on the T -equivariant K -theory $K_T(G/P)$ of the flag variety G/P , where G is a semisimple complex algebraic group, P is a parabolic subgroup and T is a maximal torus contained in P . In particular, we make a conjecture about a positivity phenomenon in $K_T(G/P)$ for the product of two basis elements written in terms of the basis of $K_T(G/P)$ given by the dual of the structure sheaf (of Schubert varieties) basis. (For the full flag variety G/B , this dual basis is closely related to the basis given by Kostant-Kumar.) This conjecture is parallel to (but different from) the conjecture of Griffeth-Ram for the structure constants of the product in the structure sheaf basis. We give explicit expressions for the product in the T -equivariant K -theory of projective spaces in terms of these bases. In particular, we establish our conjecture and the conjecture of Griffeth-Ram in this case.

1. INTRODUCTION

Let X denote the partial flag variety G/P , where G is a complex semisimple simply-connected algebraic group and P is a parabolic subgroup of G containing a fixed Borel subgroup B . The group B acts with finitely many orbits on X , and the closures of these orbits (called the Schubert varieties) are indexed by W^P , the set of minimal length coset representatives of W/W_P (where W is the Weyl group of G and W_P is the Weyl group of P); the Schubert variety corresponding to $w \in W^P$ is denoted X_w^P . The Poincaré duals of the fundamental classes $[X_w^P]$ (called the Schubert classes) form a basis for the cohomology ring $H^*(X)$. The structure constants of the multiplication in $H^*(X)$ with respect to this basis have long been known to be non-negative.

This positivity result has been generalized in different directions. If T is a maximal torus of B , then the equivariant cohomology ring $H_T^*(X)$ is a free module over $H_T^*(\text{pt})$, the equivariant cohomology ring of a point, again with a basis consisting of Schubert classes. As proved by Graham [Gra01], the structure constants in this basis again have

a positivity property generalizing the non-equivariant positivity. Similarly, the Grothendieck group $K(X)$ has a basis consisting of classes of structure sheaves $[\mathcal{O}_{X_w^P}]$ of Schubert varieties. As proved by Brion [Bri02], the structure constants of the multiplication in $K(X)$ have a predictable alternating sign behavior, which we will refer to as a *positivity property*.

The positivity in $H_T^*(X)$ and the positivity in $K(X)$ each imply the positivity in $H^*(X)$. Our aim in this paper is to discuss a positivity property for the multiplication in the T -equivariant Grothendieck group $K_T(X)$ encompassing the positivity both in $H_T(X)$ and $K(X)$. One subtlety in the T -equivariant K -theory is that $K_T(X)$ has two natural quite different bases: the basis consisting of classes of structure sheaves of Schubert varieties (called the *structure sheaf* basis), and the dual basis with respect to the natural pairing on $K_T(X)$ (called the *dual structure sheaf* basis).

Surprisingly, both the structure sheaf basis and the dual structure sheaf basis of $K_T(X)$ seem to exhibit the positivity phenomenon. Let $R(T)$ denote the representation ring of T , which is a free abelian group with basis consisting of the characters e^λ . Let Δ denote the set of roots of $\text{Lie } G$ with respect to $\text{Lie } T$, and Δ^+ the set of positive roots (chosen so that these are the roots of $\text{Lie } B$). Let $\{\xi_P^w\}$ denote the dual basis to the $R(T)$ -basis $\{[\mathcal{O}_{X_u^P}]\}$ of $K_T(X)$. Write

$$[\mathcal{O}_{X_u^P}][\mathcal{O}_{X_v^P}] = \sum_{w \in W^P} b_{u,v}^w(P) [\mathcal{O}_{X_w^P}],$$

and

$$\xi_P^u \xi_P^v = \sum_{w \in W^P} p_{u,v}^w(P) \xi_P^w,$$

for (unique) elements $b_{u,v}^w(P)$ and $p_{u,v}^w(P)$ of $R(T)$. Griffeth and Ram conjectured a positivity property for the coefficients $b_{u,v}^w(P)$. Specifically, their conjecture asserts that

$$(1) \quad (-1)^{\dim(X) + \ell(u) + \ell(v) + \ell(w)} b_{u,v}^w(P) \in \mathbb{Z}_+[e^\beta - 1]_{\beta \in \Delta^+}$$

(see Conjecture 3.10 and Remark 3.11). The validity of this conjecture for $P = B$ implies its validity for any P (cf. Proposition 3.12).

In this paper we conjecture that the coefficients $p_{u,v}^w(P)$ also exhibit the following positivity:

$$(2) \quad (-1)^{\ell(u) + \ell(v) + \ell(w)} p_{u,v}^w(P) \in \mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}$$

(see Conjecture 3.1). It is not clear if the validity of the conjecture for $P = B$ implies that for any P . On the other hand, this conjecture is compatible with the inclusion of flag varieties associated to Levi

subgroups (see Proposition 3.3). Although the coefficients $b_{u,v}^w(P)$ and $p_{u,v}^w(P)$ are related (see Propositions 4.1 and 4.3), it is not clear if one conjecture implies the other.

The non-equivariant analogues of both these conjectures hold. More precisely, if $F : R(T) \rightarrow \mathbb{Z}$ is the forgetful map (sending each e^λ to 1), then

$$(-1)^{\dim(X)+\ell(u)+\ell(v)+\ell(w)} F(b_{u,v}^w(P)) \geq 0$$

and

$$(-1)^{\ell(u)+\ell(v)+\ell(w)} F(p_{u,v}^w(P)) \geq 0.$$

The first inequality is proved in [Bri02]. The second inequality can be easily deduced from [Bri02, Theorem 1] (see Remark 3.7). In fact, we conjecture that an equivariant generalization of [Bri02, Theorem 1] holds: Let T' be a subtorus of T . If $Y \subset X$ is a T' -stable irreducible subvariety with rational singularities, and we write

$$[\mathcal{O}_Y] = \sum_{w \in W^P} a_w^Y [\mathcal{O}_{X_w^P}],$$

then

$$(-1)^{\operatorname{codim} Y + \operatorname{codim} X_w^P} a_w^Y \in \mathbb{Z}_+[e_{|T'}^{-\beta} - 1]_{\beta \in \Delta^+}$$

(see Conjecture 7.1). By Proposition 3.6, this would imply Conjecture 3.1.

The main purpose of this paper is to prove some results giving evidence for Conjectures 3.1 and 7.1. The most substantial results are explicit formulas for the coefficients $p_{u,v}^w(P)$ and $b_{u,v}^w(P)$ in case $X = \mathbb{P}^n$ (see Theorems 6.7 and 6.14). From this theorem we deduce recurrence relations (Theorems 6.11 and 6.17) for the coefficients $p_{u,v}^w(P)$ and $b_{u,v}^w(P)$ which imply Conjectures 3.1 and 3.10 for $X = \mathbb{P}^n$. We also prove that in the case $P = B$, Conjecture 3.1 holds for the coefficients $p_{u,e}^w$ and $p_{u,s}^w$, where e and s are (respectively) the identity element of W and a simple reflection (see Proposition 3.8 and Remark 3.9). We verify Conjecture 7.1 in the case Y is any opposite Schubert variety X_P^w (cf. Proposition 7.6 and Remark 7.7(a)).

A secondary purpose of this paper is to collect various results relating different bases of $K_T(X)$, and relations among the structure constants in these bases. Among the natural bases of $K_T(X)$ are the structure sheaf basis, the dual structure sheaf basis, and the basis of the dualizing sheaves of Schubert varieties. Also, one can take opposite Schubert varieties in place of Schubert varieties. The positivity conjectures have different formulations in terms of these different bases. We describe some of the relations between these bases and structure constants, in

the hope that this paper will serve as a useful reference for other workers in this area.

The contents of the paper are as follows. Section 1 lays down the basic notation. Section 2 contains some preliminary results on $K_T(G/P)$. In particular, it identifies the dual structure sheaf basis and also the basis of the dualizing sheaves of Schubert varieties (cf. Propositions 2.1 and 2.2). Section 3 contains the statement of our positivity conjecture (Conjecture 3.1). We prove the conjecture for the coefficients $p_{u,e}^w$ and $p_{u,s}^w$ (for any simple reflection s) in the case $P = B$ (cf. Proposition 3.8 and Remark 3.9). We observe that the conjecture has also been verified by an explicit calculation for any rank 2 group in the case $P = B$. By a result of Brion, the nonequivariant analogue of this conjecture holds (Remark 3.7). This section also contains the positivity conjecture of Griffeth and Ram (cf. Conjecture 3.10) and its equivalent reformulation in terms of the dualizing sheaves (cf. Proposition 3.13). It is shown that the validity of the Griffeth-Ram conjecture for $P = B$ implies its validity for any P (cf. Proposition 3.12). Section 4 proves some relations between the structure constants with respect to the structure sheaf basis and the dual structure sheaf basis (cf. Propositions 4.1 and 4.3). Section 5 proves that the structure constants with respect to either basis in the case $P = B$ lie in the subring $\mathbb{Z}[e^{-\beta} - 1]_{\beta \in \Delta^+}$ of $R(T)$ (cf. Theorem 5.1 and Corollary 5.2). Section 6 contains the explicit formula for the structure constants in the case $X = \mathbb{P}^n$ in the dual structure sheaf basis, and the recurrence relation implying Conjecture 3.1 in this case (cf. Theorems 6.7, 6.11 and 6.12). Similar results are also obtained in the structure sheaf basis. Section 7 contains our more general conjecture asserting the positivity of the coefficients of the class of the structure sheaf of a T' -stable subvariety Y of G/P with rational singularities written in terms of the structure sheaf basis (cf. Conjecture 7.1), for any subtorus $T' \subset T$. We prove this conjecture in the special case where Y is any opposite Schubert variety in any G/P (cf. Proposition 7.6 and Remark 7.7(a)).

We thank M. Brion for some helpful conversations. The first author was supported by the grant no. DMS-0403838 from NSF and the second author was supported by the FRG grant no. DMS-0554247 from NSF.

1.1. Definitions and notation. We work with schemes over the ground field of complex numbers.

Let X be a smooth algebraic variety with an action of a torus T . Let $K_T(X)$ denote the Grothendieck group of T -equivariant coherent sheaves on X ; because X is smooth, $K_T(X)$ may be identified with the Grothendieck group of T -equivariant vector bundles on X . Thus,

$K_T(X)$ is a ring; we will sometimes write the multiplication in $K_T(X)$ using the notation of tensor product. The class in $K_T(X)$ of a T -equivariant coherent sheaf \mathcal{F} will be denoted by $[\mathcal{F}]$. In particular, if $Y \subset X$ is a T -stable closed subscheme, then the structure sheaf of Y defines a class $[\mathcal{O}_Y]$ in $K_T(X)$; if Y is Cohen-Macaulay, then its dualizing sheaf ω_Y defines a class $[\omega_Y]$ in $K_T(X)$. Let $*$: $K_T(X) \rightarrow K_T(X)$ denote the standard involution taking a vector bundle to its dual and e^λ to $e^{-\lambda}$. If $r \in R(T)$, we will sometimes write \bar{r} for $*r$, where $R(T)$ is the representation ring of T . If $Y \supset Z$ are closed T -stable subschemes of X , then $\mathcal{O}_Y(-Z)$ is the ideal sheaf of Z in Y . Thus, viewed as an element of $K_T(X)$, $[\mathcal{O}_Y(-Z)] = [\mathcal{O}_Y] - [\mathcal{O}_Z]$.

Recall that $R(T)$ is a free abelian group (freely) generated by the characters e^λ of T . If V is any representation of T , we write $\text{ch } V$ for the corresponding element of $R(T)$ (a linear combination of e^λ). The group $K_T(X)$ is an $R(T)$ -module. If X is proper, and \mathcal{F} is a T -equivariant coherent sheaf on X , write $h^i(X, \mathcal{F}) = \text{ch } H^i(X, \mathcal{F})$ and

$$\chi(X, \mathcal{F}) := \sum_{p \geq 0} (-1)^p \text{ch } H^p(X, \mathcal{F}) \in R(T).$$

We extend this definition to define $\chi(X, \gamma)$ for any $\gamma \in K_T(X)$. We write $\bar{h}^i(X, \mathcal{F}) = *h^i(X, \mathcal{F})$ and $\bar{\chi}(X, \gamma) = *\chi(X, \gamma)$. For X proper, there is a pairing

$$\langle \cdot, \cdot \rangle : K_T(X) \otimes_{R(T)} K_T(X) \rightarrow R(T)$$

given by

$$\langle v_1, v_2 \rangle = \chi(X, v_1 \otimes v_2).$$

If \mathcal{F} is supported on a T -stable subscheme Y , then, viewing \mathcal{F} as a sheaf on Y , we have $\chi(X, \mathcal{F}) = \chi(Y, \mathcal{F})$.

Let G be a semisimple connected simply-connected complex algebraic group. For the rest of the paper, T will denote a maximal torus of G . Let B be a Borel subgroup of G containing T and, as above, let Δ denote the set of roots and Δ^+ the set of positive roots, chosen so that the roots of $\text{Lie } B$ are positive. Let $\{\alpha_1, \dots, \alpha_\ell\} \subset \Delta^+$ denote the simple roots, and let s_i denote the simple reflection corresponding to α_i . Let $Q^+ := \sum_i \mathbb{Z}_+ \alpha_i$. Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let B^- denote the Borel subgroup of G such that $B \cap B^- = T$.

Let $P \supset B$ be a (standard) parabolic subgroup of G and let W_P be its Weyl group. Let W^P be the set of the minimal length coset representatives in W/W_P . For $w \in W^P$, let X_w^P (respectively, X_P^w) be the Schubert variety (respectively, the opposite Schubert variety)

defined by

$$\begin{aligned} X_w^P &= \overline{BwP/P} \subset G/P \\ X_P^w &= \overline{B^-wP/P} \subset G/P. \end{aligned}$$

(Here and elsewhere we use the same notation for elements of W and lifts of those elements to G .) Set

$$\partial X_w^P = \bigsqcup_{\substack{v \in W^P \\ v < w}} BvP/P,$$

and

$$\partial X_P^w = \bigsqcup_{\substack{v \in W^P \\ v > w}} B^-vP/P.$$

Except in Section 6, we will abbreviate X_w^B and X_P^w by X_w and X^w , respectively. If λ is a character of P and \mathbb{C}_λ is the corresponding 1-dimensional representation of P , let $\mathcal{L}(\lambda)$ denote the line bundle $G \times_P \mathbb{C}_{\lambda^{-1}}$ on G/P .

Given an element $\gamma \in K_T(G/B)$, we write $\gamma(w)$ for the pullback of γ to $K_T(\{wB\}) = R(T)$.

2. PRELIMINARY RESULTS ON $K_T(G/P)$

Recall that X_w^P and X_P^w are Cohen-Macaulay (cf. [BK05, Cor. 3.4.4]) and hence their dualizing sheaves $\omega_{X_w^P}$ and $\omega_{X_P^w}$ make sense.

It is well known that $\{[\mathcal{O}_{X_w^P}]\}_{w \in W^P}$ is a $R(T)$ -basis of $K_T(G/P)$, and so is $\{[\mathcal{O}_{X_P^w}]\}_{w \in W^P}$. For any $w \in W^P$, set $\xi_P^w = [\mathcal{O}_{X_P^w}(-\partial X_P^w)] \in K_T(G/P)$.

The next proposition is known and has been observed for example by Knutson (see [Buc02, Section 8]).

Proposition 2.1. *For any $v, w \in W^P$,*

$$\langle [\mathcal{O}_{X_w^P}], \xi_P^v \rangle = \delta_{v,w},$$

i.e., $\{[\mathcal{O}_{X_w^P}]\}_{w \in W^P}$ and $\{\xi_P^w\}_{w \in W^P}$ are dual bases under the above pairing.

Proof. Since the intersections $X_w^P \cap X_P^v$ and $X_w^P \cap \partial X_P^v$ are proper (∂X_P^v is also Cohen-Macaulay since it is of pure codimension 1 in the Cohen-Macaulay variety X_P^v), we get (by [Bri02, Lemma 1])

$$\langle [\mathcal{O}_{X_w^P}], \xi_P^v \rangle = \chi(G/P, \mathcal{O}_{X_w^P \cap X_P^v}(-X_w^P \cap \partial X_P^v)).$$

By [BL03, Proposition 1],

$$\chi(\mathcal{O}_{X_w^P \cap X_P^v}) = 1 \text{ (or } 0)$$

according as

$$X_w^P \cap X_P^v \neq \emptyset \text{ (or } X_w^P \cap X_P^v = \emptyset \text{)}.$$

Similarly,

$$\chi(\mathcal{O}_{X_w^P \cap \partial X_P^v}) = 1 \text{ (or } 0 \text{)}$$

according as

$$X_w^P \cap \partial X_P^v \neq \emptyset \text{ (or } X_w^P \cap \partial X_P^v = \emptyset \text{)}.$$

Now,

$$\begin{aligned} X_w^P \cap X_P^v \neq \emptyset &\Leftrightarrow w \geq v \quad \text{and} \\ X_w^P \cap \partial X_P^v \neq \emptyset &\Leftrightarrow \text{there exists a } \theta \in W^P \text{ such that } w \geq \theta > v \\ &\Leftrightarrow w > v. \end{aligned}$$

Combining the above, we get the proposition. \square

Let $\{\tau^w\}_{w \in W}$ be the Kostant-Kumar $R(T)$ -basis of $K_T(G/B)$ (cf. [KK90, Remark 3.14]). We abbreviate ξ_B^w by ξ^w (as noted above, X_w^B and X_w^w are abbreviated as X_w and X^w , respectively).

The next proposition gives some of the relations between various T -equivariant sheaves on G/B and between elements of $K_T(G/B)$.

Proposition 2.2. *For any $w \in W$*

- (a) $\omega_{X_w} \simeq e^{-\rho} \mathcal{L}(-\rho) \otimes \mathcal{O}_{X_w}(-\partial X_w)$ as T -equivariant sheaves.
- (b) $\omega_{X^w} \simeq e^{\rho} \mathcal{L}(-\rho) \otimes \mathcal{O}_{X^w}(-\partial X^w)$ as T -equivariant sheaves.
- (c) $*\tau^w = \xi^{w^{-1}} = e^{-\rho} [\mathcal{L}(\rho)] [\omega_{X^{w^{-1}}}]$, as elements of $K_T(G/B)$.
- (d) $e^{\rho} [\mathcal{L}(\rho)] (*\tau^w) = (-1)^{\ell(w)} * [\mathcal{O}_{X^{w^{-1}}}]$.

Proof. By [Ram87, Theorem 4.2], as non-equivariant sheaves,

$$\omega_{X_w} \simeq \mathcal{L}(-\rho) \otimes \mathcal{O}_{X_w}(-\partial X_w).$$

We now determine ω_{X_w} as a T -equivariant sheaf. Since BwB/B is a smooth open subset of X_w , $\omega_{X_w}|_{(BwB/B)}$ is the canonical line bundle. The fiber of ω_{X_w} at the T -fixed point $wB \in BwB/B$ as a T -module is given by the character $|\Delta^- \cap w\Delta^+| = \sum_{\alpha \in \Delta^- \cap w\Delta^+} \alpha = w\rho - \rho$. The fiber of $\mathcal{L}(-\rho)$ at wB has weight $w\rho$ and clearly the fiber of $\mathcal{O}_{X_w}(-\partial X_w)$ at wB has weight 0. Combining the above, we get (a). (Here we have used the fact that on a reflexive sheaf \mathcal{S} of rank 1 on an irreducible projective T -variety X , there exists at most one T -equivariant structure such that the induced T -module structure on the stalk of \mathcal{S} at a T -fixed point $x_0 \in X$ is trivial.)

The proof of (b) is similar.

By [KK90, Proposition 3.39], for any $v, w \in W$,

$$(3) \quad \chi(X_{v^{-1}}, * \tau^w) = \langle \mathcal{O}_{X_{v^{-1}}}, * \tau^w \rangle = \delta_{v,w}.$$

By the preceding proposition, this implies that $* \tau^w = \xi^{w^{-1}}$, proving the first equality of (c). The second equality of (c) follows from (b) and the definition of ξ^w . By [Bri02, §2] (which holds equivariantly), for any closed T -stable Cohen-Macaulay subvariety $Y \subset G/B$, we have

$$*[\mathcal{O}_Y] = (-1)^{\text{codim } Y} [\omega_Y] \cdot *[\omega_{G/B}].$$

Part (d) follows by combining (c) with this equation for $Y = X^{w^{-1}}$, using the fact that $\omega_{G/B} \cong \mathcal{L}(-2\rho)$. \square

3. A POSITIVITY CONJECTURE FOR $K_T(G/P)$

3.1. Positivity in the dual Schubert basis. We make the following conjecture concerning the multiplication in $K_T(G/P)$ in terms of the basis $\{\xi_P^w\}$.

Conjecture 3.1. For any (standard) parabolic subgroup P and any $u, v \in W^P$, express

$$(4) \quad \xi_P^u \xi_P^v = \sum_{w \in W^P} p_{u,v}^w(P) \xi_P^w,$$

for some (unique) $p_{u,v}^w(P) \in R(T)$. Then,

$$(-1)^{\ell(u)+\ell(v)+\ell(w)} p_{u,v}^w(P) \in \mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+},$$

where the notation $\mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}$ means polynomials in $\{e^{-\beta} - 1\}_{\beta \in \Delta^+}$ with coefficients in \mathbb{Z}_+ .

We will write simply $p_{u,v}^w$ for $p_{u,v}^w(B)$.

Remarks 3.2. (a) By an explicit case by case calculation, we have verified the validity of the above conjecture for $P = B$ and any rank-2 group G .

(b) We show the validity of our conjecture when $G = SL_{n+1}$ and P is the standard maximal parabolic subgroup corresponding to the first node (so that $G/P = \mathbb{P}^n$) in Section 6.

The next proposition gives a relation between structure constants under the inclusion of flag varieties associated to Levi subgroups. In the special case where the flag varieties are projective spaces, this result also follows from our explicit calculation of the structure constants (see Corollary 6.10).

Proposition 3.3. *Let G, P, T be as in the above conjecture and let L be the Levi subgroup of P containing T . Let Q be a standard parabolic subgroup of G contained in P and let $Q_L := L \cap Q$ be the corresponding parabolic subgroup of L . Then, for any $u, v, w \in (W_P)^{Q_L}$,*

$$p_{u,v}^w(Q_L) = p_{u,v}^w(Q),$$

where $p_{u,v}^w(Q_L)$ are the structure constants for the flag variety L/Q_L . (Observe that $(W_P)_{Q_L}$ can canonically be identified with W_Q and $(W_P)^{Q_L}$ is canonically embedded in W^Q .)

Proof. Observe that the canonical inclusion $i : L/Q_L \hookrightarrow G/Q$ takes the Schubert variety $X_w^{Q_L} \subset L/Q_L$ isomorphically onto the Schubert variety $X_w^Q \subset G/Q$, for any $w \in (W_P)^{Q_L}$. For $w \in W^Q$, we claim that $i^*(\xi_Q^w)$ equals $\xi_{Q_L}^w$ if $w \in (W_P)^{Q_L}$, and is 0 otherwise. Indeed, for $u \in (W_P)^{Q_L}$,

$$\chi(X_u^{Q_L}, i^*(\xi_Q^w)) = \chi(X_u^Q, \xi_Q^w) = \delta_{u,w},$$

proving the claim.

For $u, v \in (W_P)^{Q_L}$, we have

$$\xi_{Q_L}^u \xi_{Q_L}^v = \sum_{w \in (W_P)^{Q_L}} p_{u,v}^w(Q_L) \xi_{Q_L}^w.$$

On the other hand, since i^* is a ring homomorphism, we have

$$\xi_{Q_L}^u \xi_{Q_L}^v = i^*(\xi_Q^u \xi_Q^v) = i^*\left(\sum_{w \in W^Q} p_{u,v}^w(Q) \xi_Q^w\right) = \sum_{w \in (W_P)^{Q_L}} p_{u,v}^w(Q) \xi_{Q_L}^w.$$

Comparing these two expressions, we get the proposition. \square

Lemma 3.4. *Let $\pi : G/B \rightarrow G/P$ denote the projection. Then*

$$\pi^*(\xi_P^v) = \sum_{u \in vW_P} \xi^u, \text{ for any } v \in W^P.$$

Proof. We have $\langle \pi^*(\xi_P^v), [\mathcal{O}_{X_u}] \rangle = \langle \xi_P^v, \pi_*[\mathcal{O}_{X_u}] \rangle$. Further, $\pi_*[\mathcal{O}_{X_u}] = [\mathcal{O}_{\pi(X_u)}]$ by [BK05, Theorem 3.3.4(a)]. Thus, by Proposition 2.1, $\langle \xi_P^v, \pi_*[\mathcal{O}_{X_u}] \rangle$ is 0 unless $\pi(X_u) = X_v^P$, and this holds if and only if $u \in vW_P$. The lemma follows from this, together with Proposition 2.1 applied to the case of G/B . \square

Unlike Conjecture 3.10 below due to Griffeth-Ram, the validity of the above conjecture for $P = B$ does not seem to give the validity of the conjecture for an arbitrary (standard) parabolic P . In fact, we have the following proposition relating the structure constants for P and B .

Proposition 3.5. *For any $u, v, w \in W^P$,*

$$p_{u,v}^w(P) = \sum_{\substack{u' \in uW_P \\ v' \in vW_P}} p_{u',v'}^w(B).$$

Proof. Since

$$\xi_P^u \xi_P^v = \sum_{w \in W^P} p_{u,v}^w(P) \xi_P^w,$$

taking π^* and using the above lemma, we get

$$\sum_{\substack{u' \in uW_P \\ v' \in vW_P}} \xi^{u'} \xi^{v'} = \sum_{w \in W^P} (p_{u,v}^w(P) \sum_{w' \in wW_P} \xi^{w'}),$$

i.e.,

$$\sum_{\theta \in W} \sum_{\substack{u' \in uW_P \\ v' \in vW_P}} p_{u',v'}^\theta(B) \xi^\theta = \sum_{w \in W^P} \sum_{w' \in wW_P} p_{u,v}^w(P) \xi^{w'}.$$

Equating the coefficients from the two sides, we get the proposition. \square

Let D be the diagonal map $G/P \rightarrow G/P \times G/P$. This, of course, induces the push-forward map

$$D_* : K_T(G/P) \longrightarrow K_T(G/P) \otimes_{R(T)} K_T(G/P),$$

$$D_*[\mathcal{F}] = \sum_{p \geq 0} (-1)^p [R^p D_* \mathcal{F}],$$

and also the pull-back (product) map

$$D^* : K_T(G/P) \otimes_{R(T)} K_T(G/P) \longrightarrow K_T(G/P).$$

Here we have identified $K_T(G/P \times G/P)$ with $K_T(G/P) \otimes_{R(T)} K_T(G/P)$ (cf. [CG97, Theorem 5.6.1]). The next proposition gives another description of the coefficients $p_{u,v}^w(P)$.

Proposition 3.6. *For any $u, v, w \in W^P$,*

$$D_*[\mathcal{O}_{X_w^P}] = \sum_{u,v \in W^P} p_{u,v}^w(P) [\mathcal{O}_{X_u^P}] \boxtimes [\mathcal{O}_{X_v^P}].$$

Proof. This follows from functorial properties of K -theory. To see this, for any space Y , write π_Y for the projection from Y to a point. Write $X = G/P$. By definition, $\chi(X, \mathcal{F}) = \pi_{X*}(\mathcal{F})$. By definition of the

coefficients $p_{u,v}^w(P)$ and Proposition 2.1,

$$\begin{aligned} p_{u,v}^w(P) &= \pi_{X*}(\xi_P^u \xi_P^v \otimes [\mathcal{O}_{X_w^P}]) \\ &= \pi_{X*}(D^*(\xi_P^u \boxtimes \xi_P^v) \otimes [\mathcal{O}_{X_w^P}]) \\ &= (\pi_{X \times X})_* D_*(D^*(\xi_P^u \boxtimes \xi_P^v) \otimes [\mathcal{O}_{X_w^P}]) \\ &= (\pi_{X \times X})_* ((\xi_P^u \boxtimes \xi_P^v) \otimes D_*[\mathcal{O}_{X_w^P}]). \end{aligned}$$

Since $\{\xi_P^u \boxtimes \xi_P^v\}$ and $\{[\mathcal{O}_{X_u^P}] \boxtimes [\mathcal{O}_{X_v^P}]\}$ are dual bases of $K_T(X \times X)$, the lemma follows. \square

Remark 3.7. The non-equivariant analogue of the preceding proposition holds with the same proof. Combining this with [Bri02, Theorem 1], we see that the structure constants $F(p_{u,v}^w(P))$ for the non-equivariant multiplication in the basis $\{\xi_P^u\}_u$ (cf. equation (4) of Conjecture 3.1); here $F : R(T) \rightarrow \mathbb{Z}$ is the forgetful map) satisfy

$$(-1)^{\ell(w)+\ell(u)+\ell(v)} F(p_{u,v}^w(P)) \in \mathbb{Z}_+.$$

For any subset $S \subset \{1, \dots, \ell\}$ (including $S = \emptyset$), let W_S be the subgroup of W generated by the simple reflections $\{s_i, i \in S\}$. Recall that $Q^+ := \sum_i \mathbb{Z}_+ \alpha_i$.

Proposition 3.8. *For any $u, w \in W$, and any $S \subset \{1, \dots, \ell\}$, we have*

$$(-1)^{\ell(w)+\ell(u)} \sum_{v \in W_S} p_{u,v}^w \in \sum_{\beta \in Q^+} \mathbb{Z}_+ e^{-\beta}.$$

In particular,

$$(-1)^{\ell(w)+\ell(u)} p_{u,e}^w \in \mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}.$$

Proof. As in Proposition 3.6, write

$$D_*[\mathcal{O}_{X_w}] = \sum_{u,v \in W} p_{u,v}^w [\mathcal{O}_{X_u}] \boxtimes [\mathcal{O}_{X_v}].$$

Pairing this with $\xi^u \boxtimes \mathcal{L}(-\rho_S)$, we get

$$(5) \quad \chi(X_w, \xi^u \otimes \mathcal{L}(-\rho_S)) = \sum_{v \in W} p_{u,v}^w \chi(X_v, \mathcal{L}(-\rho_S)),$$

where ρ_i is the i -th fundamental weight and $\rho_S := \sum_{i \notin S} \rho_i$.

We claim that, for any $v \notin W_S$,

$$(6) \quad H^i(X_v, \mathcal{L}(-\rho_S)) = 0 \text{ for all } i \geq 0.$$

Let $P = P_S$ be the parabolic subgroup corresponding to the subset S , i.e., the Levi subgroup of P_S containing T has for its simple roots

$\{\alpha_i\}_{i \in S}$. Then, the line bundle $\mathcal{L}(-\rho_S)$ is the pull-back of a line bundle on G/P . Moreover, by [BK05, Theorem 3.3.4],

$$(7) \quad H^i(X_v, \mathcal{L}(-\rho_S)) \cong H^i(X_{v'}, \mathcal{L}(-\rho_S)),$$

where v' is the coset representative of minimal length in the coset vW_S . Take s_j , $j \notin S$, such that $v's_j < v'$. Then, the standard projection $\pi : X_{v'} \rightarrow X_{v'}^{P_j}$ is a \mathbb{P}^1 -fibration and $\mathcal{L}(-\rho_S)$ has degree -1 along the fibers of π , where $P_j = P_{\{j\}}$. Hence, $R^i \pi_* \mathcal{L}(-\rho_S) = 0$ for all i ; (6) follows from this and the Leray spectral sequence together with (7).

For $v \in W_S$, by (7),

$$H^i(X_v, \mathcal{L}(-\rho_S)) \cong H^i(X_e, \mathcal{L}(-\rho_S)).$$

Thus,

$$\begin{aligned} H^i(X_v, \mathcal{L}(-\rho_S)) &= 0 \quad \text{if } i > 0 \text{ and} \\ \text{ch } H^0(X_v, \mathcal{L}(-\rho_S)) &= e^{\rho_S}. \end{aligned}$$

Thus, by (5),

$$\chi(X_w, \xi^u \otimes \mathcal{L}(-\rho_S)) = \left(\sum_{v \in W_S} p_{u,v}^w \right) e^{\rho_S},$$

i.e.,

$$(8) \quad \sum_{v \in W_S} p_{u,v}^w = e^{-\rho_S} \chi(X_w, \xi^u \otimes \mathcal{L}(-\rho_S)).$$

Now,

$$\chi(X_w, \xi^u \otimes \mathcal{L}(-\rho_S)) = \chi(X_w \cap X^u, \mathcal{L}(-\rho_S)(-X_w \cap \partial X^u)).$$

By [Bri02, Theorem 4], this equals

$$(9) \quad (-1)^{\ell(u)+\ell(w)} w_o \left(* \text{ch} \left(H^0(X_{w_o u} \cap X^{w_o w}, \mathcal{L}(\rho_S)(-X_{w_o u} \cap \partial X^{w_o w})) \right) \right),$$

where w_o is the longest element of W . (Brion's result is stated non-equivariantly, but if we change his duality formula to the following:

$$c_v^w(\lambda) = (-1)^{\ell(v)+\ell(w)} w_o \cdot (*c_{w_o w}^{w_o v}(-\lambda)),$$

then it remains true T -equivariantly by a similar proof.) By [BL03, Proposition 1], the restriction map

$$H^0(G/B, \mathcal{L}(\rho_S)) \longrightarrow H^0(X_{w_o u} \cap X^{w_o w}, \mathcal{L}(\rho_S))$$

is surjective. Also,

$$H^0(X_{w_o u} \cap X^{w_o w}, \mathcal{L}(\rho_S)(-X_{w_o u} \cap \partial X^{w_o w})) \subset H^0(X_{w_o u} \cap X^{w_o w}, \mathcal{L}(\rho_S)).$$

Since $H^0(G/B, \mathcal{L}(\rho_S))$ is the irreducible G -module with highest weight $-w_o\rho_S$, we see that

$$(10) \quad e^{-\rho_S} w_o \left(* \text{ch} \left(H^0(X_{w_o u} \cap X^{w_o w}, \mathcal{L}(\rho_S)(-X_{w_o u} \cap \partial X^{w_o w})) \right) \right) \in \sum_{\beta \in Q^+} \mathbb{Z}_+ e^{-\beta}.$$

Combining (8)-(10), we get the lemma. \square

Remark 3.9. Comparing the expression obtained in the above proof for $p_{u,s_i}^w + p_{u,e}^w$ and $p_{u,e}^w$ (for any simple reflection s_i), it can be shown that

$$(-1)^{\ell(u)+\ell(w)+1} p_{u,s_i}^w \in \mathbb{Z}_+ [e^{-\beta} - 1]_{\beta \in \Delta^+}.$$

3.2. Positivity in the structure sheaf basis. We recall below the conjecture of Griffeth and Ram on the nonnegativity of the product in the structure sheaf basis. They verified their conjecture for rank-2 groups by an explicit case by case calculation. In this section we prove that the validity of their conjecture for $P = B$ implies its validity for every (standard) parabolic subgroup P , and give an equivalent formulation of their conjecture in terms of dualizing sheaves.

Conjecture 3.10. For any standard parabolic subgroup P and $u, v \in W^P$, express

$$(11) \quad [\mathcal{O}_{X_P^u}][\mathcal{O}_{X_P^v}] = \sum_{w \in W^P} c_{u,v}^w(P) [\mathcal{O}_{X_P^w}] \in K_T(G/P),$$

for some (unique) $c_{u,v}^w(P) \in R(T)$.

Then,

$$(-1)^{\ell(u)+\ell(v)+\ell(w)} c_{u,v}^w(P) \in \mathbb{Z}_+ [e^{-\beta} - 1]_{\beta \in \Delta^+}.$$

Remark 3.11. Express

$$[\mathcal{O}_{X_P^u}][\mathcal{O}_{X_P^v}] = \sum_{w \in W^P} b_{u,v}^w(P) [\mathcal{O}_{X_P^w}].$$

Then, an equivalent formulation of the above conjecture asserts that

$$(-1)^{\dim(G/P)+\ell(u)+\ell(v)+\ell(w)} b_{u,v}^w(P) \in \mathbb{Z}_+ [e^{\beta} - 1]_{\beta \in \Delta^+}.$$

Moreover, an argument similar to the proof of Proposition 3.6 shows that the structure constants $b_{u,v}^w(P)$ are described by the equation:

$$D_*(\xi_P^w) = \sum_{u,v \in W^P} b_{u,v}^w(P) \xi_P^u \boxtimes \xi_P^v.$$

Proposition 3.12. *We have*

$$(12) \quad c_{u,v}^w(P) = c_{u,v}^w(B), \text{ for any } u, v, w \in W^P.$$

Hence, the validity of the above Conjecture 3.10 for $P = B$ implies its validity for every (standard) parabolic subgroup P .

Proof. Let $\pi : G/B \rightarrow G/P$ be the standard projection. Since π is a T -equivariant smooth morphism, for any T -stable closed subvariety $Z \subseteq G/P$,

$$\pi^*[\mathcal{O}_Z] = [\mathcal{O}_{\pi^{-1}(Z)}].$$

Thus,

$$(13) \quad \begin{aligned} \pi^*[\mathcal{O}_{X_P^w}] &= [\mathcal{O}_{\pi^{-1}(X_P^w)}] \\ &= [\mathcal{O}_{X^w}], \end{aligned}$$

since $B^-wP/P = w_oBw_oP/P$ and w_oP is the longest element in the W_P -orbit w_oP .

Since π^* is a ring homomorphism, (12) follows from (13). \square

As a consequence of this proposition, we will simply write $c_{u,v}^w$ for $c_{u,v}^w(P)$.

The following proposition provides an equivalent formulation of Conjecture 3.10 in terms of dualizing sheaves.

Proposition 3.13. *For any standard parabolic subgroup P and any $u, v \in W^P$, express*

$$(14) \quad [\omega_{X_P^u}] \cdot [\omega_{X_P^v}] = \sum_{w \in W^P} d_{u,v}^w(P) [\omega_{X_P^w}] [\omega_{G/P}],$$

for some (unique) $d_{u,v}^w(P) \in R(T)$. Then,

$$(15) \quad d_{u,v}^w(P) = (-1)^{\ell(u)+\ell(v)+\ell(w)} * (c_{u,v}^w).$$

In particular, Conjecture 3.10 is equivalent to the conjecture that $d_{u,v}^w(P) \in \mathbb{Z}_+[e^\beta - 1]_{\beta \in \Delta^+}$.

Moreover,

$$d_{u,v}^w(P) = d_{u,v}^w(B), \text{ for any } u, v, w \in W^P.$$

Proof. By [Bri02, §2],

$$(16) \quad *[\mathcal{O}_{X_P^v}] = (-1)^{\ell(v)} [\omega_{X_P^v}] \cdot *[\omega_{G/P}].$$

Multiply equation (14) by $*[\omega_{G/P}]^2$ to get

$$[\omega_{X_P^u}] * [\omega_{G/P}] [\omega_{X_P^v}] * [\omega_{G/P}] = \sum_{w \in W^P} d_{u,v}^w(P) [\omega_{X_P^w}] * [\omega_{G/P}].$$

By (16), the above equation reduces to

$$*[\mathcal{O}_{X_P^u}] \cdot *[\mathcal{O}_{X_P^v}] = \sum_{w \in W^P} (-1)^{\ell(u)+\ell(v)+\ell(w)} d_{u,v}^w(P) * [\mathcal{O}_{X_P^w}].$$

Comparing this with the identity (11) of Conjecture 3.10, we get

$$d_{u,v}^w(P) = (-1)^{\ell(u)+\ell(v)+\ell(w)} * (c_{u,v}^w).$$

This proves (15). \square

4. RELATIONS BETWEEN STRUCTURE SHEAF CONSTANTS

In this section we restrict to the case $P = B$, and prove two relations (Propositions 4.1 and 4.3) between the structure constants in the structure sheaf basis and the structure constants in the dual basis. However, we do not know how to use these relations to relate the two positivity conjectures 3.1 and 3.10.

Write

$$[\mathcal{L}(\rho)][\mathcal{O}_{X^\theta}] = \sum_{w \in W} d_w^\theta [\mathcal{O}_{X^w}],$$

for some (unique) $d_w^\theta \in R(T)$. The following proposition gives a relation between the structure constants $c_{u,v}^w$ and $p_{u,v}^w$.

Proposition 4.1. *For any $u, v, w \in W$,*

$$c_{u,v}^w = (-1)^{\ell(u)+\ell(v)} \sum_{\theta \in W} (-1)^{\ell(\theta)} e^\rho d_w^\theta \bar{p}_{u,v}^\theta.$$

Proof. By Proposition 2.2,

$$*\xi^w = \tau^{w^{-1}} = (-1)^{\ell(w)} e^\rho [\mathcal{L}(\rho)][\mathcal{O}_{X^w}].$$

We have

$$(*\xi^u)(* \xi^v) = \sum_{\theta \in W} \bar{p}_{u,v}^\theta (* \xi^\theta),$$

and hence

$$\begin{aligned} [\mathcal{O}_{X^u}][\mathcal{O}_{X^v}] &= (-1)^{\ell(u)+\ell(v)} \sum_{\theta \in W} (-1)^{\ell(\theta)} \bar{p}_{u,v}^\theta e^\rho [\mathcal{L}(\rho)][\mathcal{O}_{X^\theta}] \\ &= (-1)^{\ell(u)+\ell(v)} \sum_{\theta, w \in W} (-1)^{\ell(\theta)} \bar{p}_{u,v}^\theta e^\rho d_w^\theta [\mathcal{O}_{X^w}]. \end{aligned}$$

From this the proposition follows. \square

Before stating the second relation between structure constants, we compare the bases $\{\xi^v\}_{v \in W}$ and $\{[\mathcal{O}_{X^v}]\}_{v \in W}$ of $K_T(G/B)$. Let

$$\mu(v, w) = \begin{cases} (-1)^{\ell(v)+\ell(w)} & \text{if } v \leq w \\ 0 & \text{otherwise} \end{cases}$$

denote the Möbius function of the Weyl group W .

Lemma 4.2. *For any $v \in W$, write*

$$[\mathcal{O}_{X^v}] = \sum_{w \in W} e_{v,w} \xi^w.$$

Then

$$\begin{aligned} e_{v,w} &= 1, & \text{if } v \leq w \\ &= 0, & \text{otherwise.} \end{aligned}$$

Thus,

$$\xi^v = \sum_w \mu(v, w) [\mathcal{O}_{X^w}].$$

Proof. By Proposition 2.1,

$$e_{v,w} = \langle [\mathcal{O}_{X_w}], [\mathcal{O}_{X^v}] \rangle = \chi(G/B, [\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X^v}]).$$

Since $X_w \cap X^v$ is a proper intersection, by [Bri02, Lemma 1], $[\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X^v}] = [\mathcal{O}_{X_w \cap X^v}]$. Thus, $e_{v,w} = \chi(X_w \cap X^v, \mathcal{O}_{X_w \cap X^v})$. By [BL03, Proposition 1] (cf. proof of Proposition 2.1),

$$\chi(X_w \cap X^v, \mathcal{O}_{X_w \cap X^v}) = 1 \text{ (or } 0)$$

according as $X_w \cap X^v$ is nonempty (or empty), i.e.,

$$\begin{aligned} \chi(X_w \cap X^v, \mathcal{O}_{X_w \cap X^v}) &= 1, & \text{if } v \leq w \\ &= 0 & \text{otherwise.} \end{aligned}$$

This proves the first part of the lemma.

Define the matrix

$$E = (e_{v,w})_{v,w \in W}.$$

Then, by [Deo77, §3], E^{-1} is the Möbius function, i.e.,

$$(E^{-1})_{v,w} = \mu(v, w).$$

From this the second part of the lemma follows. \square

We can now state the second relation between the structure constants.

Proposition 4.3. *For any $u, v, w \in W$,*

$$(17) \quad c_{u,v}^w = (-1)^{\ell(w)} \sum_{\substack{u \leq y \\ v \leq z \\ \theta \leq w}} (-1)^{\ell(\theta)} p_{y,z}^\theta.$$

Similarly,

$$(18) \quad p_{u,v}^w = (-1)^{\ell(u)+\ell(v)} \sum_{\substack{u \leq y \\ v \leq z \\ \theta \leq w}} (-1)^{\ell(y)+\ell(z)} c_{y,z}^\theta.$$

Proof. By Lemma 4.2,

$$\begin{aligned} [\mathcal{O}_{X^u}] [\mathcal{O}_{X^v}] &= \left(\sum_y e_{u,y} \xi^y \right) \cdot \left(\sum_z e_{v,z} \xi^z \right) \\ &= \sum_{y,z} e_{u,y} e_{v,z} \xi^y \xi^z \\ &= \sum_{y,z,\theta} e_{u,y} e_{v,z} p_{y,z}^\theta \xi^\theta \\ &= \sum_{y,z,\theta,w} e_{u,y} e_{v,z} p_{y,z}^\theta \mu(\theta, w) [\mathcal{O}_{X^w}] \\ &= \sum_{\substack{u \leq y \\ v \leq z \\ \theta \leq w}} p_{y,z}^\theta (-1)^{\ell(\theta)+\ell(w)} [\mathcal{O}_{X^w}]. \end{aligned}$$

Thus, equating the coefficients in the $\{[\mathcal{O}_{X^w}]\}_w$ basis, we get

$$c_{u,v}^w = (-1)^{\ell(w)} \sum_{\substack{u \leq y \\ v \leq z \\ \theta \leq w}} (-1)^{\ell(\theta)} p_{y,z}^\theta.$$

To prove (18), write by Lemma 4.2,

$$\begin{aligned} \xi^u \xi^v &= \left(\sum_y \mu(u, y) [\mathcal{O}_{X^y}] \right) \cdot \left(\sum_z \mu(v, z) [\mathcal{O}_{X^z}] \right) \\ &= \sum_{y,z} \mu(u, y) \mu(v, z) [\mathcal{O}_{X^y}] \cdot [\mathcal{O}_{X^z}] \\ &= \sum_{y,z,\theta} \mu(u, y) \mu(v, z) c_{y,z}^\theta [\mathcal{O}_{X^\theta}] \\ &= \sum_{y,z,\theta,w} \mu(u, y) \mu(v, z) c_{y,z}^\theta e_{\theta,w} \xi^w. \end{aligned}$$

Thus,

$$\begin{aligned} p_{u,v}^w &= \sum_{y,z,\theta} \mu(u,y)\mu(v,z) c_{y,z}^\theta e_{\theta,w} \\ &= \sum_{\substack{u \leq y \\ v \leq z \\ \theta \leq w}} (-1)^{\ell(u)+\ell(y)+\ell(v)+\ell(z)} c_{y,z}^\theta. \end{aligned}$$

This proves the proposition. \square

5. MULTIPLICATIVE STRUCTURE CONSTANTS LIE IN $\mathbb{Z}[e^{-\beta} - 1]$

Let Z denote the center of G , and let $T' = T/Z$. The map $T \rightarrow T'$ induces an injection $R(T') \hookrightarrow R(T)$ whose image is the subring $\mathbb{Z}[e^\beta]_{\beta \in \Delta}$ of $R(T)$. Of course, $\mathbb{Z}[e^\beta]_{\beta \in \Delta} = \mathbb{Z}[e^\beta - 1]_{\beta \in \Delta}$; writing the ring in this way emphasizes the relationship with the positivity conjectures.

The main result of this section is the following theorem concerning the structure constants in the dual structure sheaf basis, in the case $P = B$.

Theorem 5.1. *With the notation as in Conjecture 3.1, for any $u, v, w \in W$, we have*

$$p_{u,v}^w \in \mathbb{Z}[e^{-\beta} - 1]_{\beta \in \Delta+}.$$

Proof. We first show that $p_{u,v}^w \in R(T')$. Because Z acts trivially on G/B , the action of T on G/B factors through the action of $T' = T/Z$. Therefore, there is a canonical map $K_{T'}(X) \rightarrow K_T(X)$ compatible with the map $R(T') \rightarrow R(T)$. By the cellular fibration lemma [CG97, Lemma 5.5.1], $K_{T'}(X)$ is free over $R(T')$ and the classes of \mathcal{O}_{X_w} in $K_{T'}(X)$ form a basis. Since the class of \mathcal{O}_{X_w} in $K_{T'}(X)$ maps to the class of the same sheaf in $K_T(X)$, and the map $K_{T'}(X) \rightarrow K_T(X)$ is a ring homomorphism, the structure constants $b_{u,v}^w$ of the multiplication in $K_T(X)$ with respect to this basis must be the images of the corresponding structure constants in $K_{T'}(X)$, and hence must lie in $R(T')$. Thus, by (18), $p_{u,v}^w$ must lie in $R(T')$ as well.

By Proposition 2.2(c),

$$(19) \quad \xi^w = * \tau^{w^{-1}}, \text{ for any } w \in W.$$

By [KK90, Proposition 2.22(h)], $p_{u,v}^w = 0$ unless $u, v \leq w$. We will prove the proposition by induction on $\ell(w)$. The proposition is true for $w = 1$ since $p_{u,v}^1 = 0$ unless $u = v = 1$. Moreover, $p_{1,1}^1 = 1$, since by [KK90, Proposition 2.22(b),(f)],

$$\tau^1 \tau^1 = \tau^1 + \sum_{w \neq e} d^w \tau^w,$$

for some $d_w \in R(T)$.

Fix $u, v \in W$ and assume by induction that $p_{u,v}^\theta \in \mathbb{Z}[e^{-\beta} - 1]$ for all $\theta < w$. Write

$$\tau^{u^{-1}} \cdot \tau^{v^{-1}} = \sum_{\theta < w} a_{u,v}^\theta \tau^{\theta^{-1}} + a_{u,v}^w \tau^{w^{-1}} + \sum_{\delta \not\leq w} a_{u,v}^\delta \tau^{\delta^{-1}}.$$

By (19),

$$(20) \quad a_{u,v}^w = \bar{p}_{u,v}^w, \text{ for any } u, v, w \in W.$$

Now,

$$(21) \quad \tau^{u^{-1}}(w^{-1})\tau^{v^{-1}}(w^{-1}) = \sum_{\theta < w} a_{u,v}^\theta \tau^{\theta^{-1}}(w^{-1}) + a_{u,v}^w \tau^{w^{-1}}(w^{-1}),$$

as $\tau^{\delta^{-1}}(w^{-1}) = 0$ for $\delta \not\leq w$ by [KK90, Proposition 2.22(b)]. By [Wil06] or [Gra02],

$$(22) \quad \tau^{\theta^{-1}}(w^{-1}) \in \mathbb{Z}[e^\beta - 1]_{\beta \in \Delta^+}.$$

Moreover, by [KK90, Proposition 2.22(b)],

$$(23) \quad \tau^{w^{-1}}(w^{-1}) = \prod_{\nu \in w\Delta^- \cap \Delta^+} (1 - e^\nu).$$

Let $x_i := e^{\alpha_i}$, $1 \leq i \leq \ell$, where $\{\alpha_1, \dots, \alpha_\ell\}$ are the simple roots. Then, $R := \mathbb{Z}[e^\beta]_{\beta \in \Delta^+}$ is the polynomial ring $\mathbb{Z}[x_1, \dots, x_\ell]$, and

$$R \subset R(T') := \mathbb{Z}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}].$$

We have proved earlier that $a_{u,v}^w$ is in $R(T')$; thus, to prove the theorem, it suffices by (20) to show that $a_{u,v}^w \in R$. By (20)–(22), and induction,

$$(24) \quad a_{u,v}^w \tau^{w^{-1}}(w^{-1}) \in R.$$

Moreover, by (23), $\tau^{w^{-1}}(w^{-1})$ is in R and does not vanish at 0, so $a_{u,v}^w$ has no pole at 0. Hence,

$$a_{u,v}^w \in \mathbb{C}[x_1, \dots, x_\ell].$$

We next show that the coefficient of each monomial in $a_{u,v}^w$ must be an integer, so that $a_{u,v}^w \in R$. Write

$$a_{u,v}^w = \sum_{\underline{d} \in \mathbb{Z}_+^\ell} c_{\underline{d}} \underline{x}^{\underline{d}}, \text{ for } c_{\underline{d}} \in \mathbb{C},$$

where $\underline{x}^{\underline{d}} := x_1^{d_1} \cdots x_\ell^{d_\ell}$ for $\underline{d} = (d_1, \dots, d_\ell)$. Choose, if possible, $c_{\underline{d}^o} \notin \mathbb{Z}$ so that $|\underline{d}^o| := d_1^o + \cdots + d_\ell^o$ is minimum with this property. Then, (24) implies that

$$\left(\sum_{|\underline{d}| \geq |\underline{d}^o|} c_{\underline{d}} \underline{x}^{\underline{d}} \right) \tau^{w^{-1}}(w^{-1}) \in R,$$

which is a contradiction, since the constant term of $\tau^{w^{-1}}(w^{-1})$ is 1. Hence, $a_{u,v}^w \in R$, as desired. \square

Corollary 5.2. *For any $u, v, w \in W$, we have that $c_{u,v}^w \in \mathbb{Z}[e^{-\beta} - 1]_{\beta \in \Delta^+}$.*

Proof. This follows by combining Theorem 5.1 and Proposition 4.3. Alternatively, it can be deduced by an argument similar to the proof of Theorem 5.1, using the formula of [Gra02] for the pullback of elements of the structure sheaf basis to T -fixed points. \square

6. POSITIVITY IN EQUIVARIANT K -THEORY OF \mathbb{P}^n

In this section we prove an explicit formula for the structure constants in the dual structure sheaf basis in case $G/P = \mathbb{P}^n$. We use this to deduce a recurrence relation which implies Conjecture 3.1 in this case. We give the analogous results for the structure sheaf basis. This section can be read independently of the previous sections, except for references to a few results.

6.1. Preliminary results on $K_T(\mathbb{P}^n)$. Let $X = \mathbb{P}^n$ with projective coordinates $[x_1, \dots, x_{n+1}]$. Thus, $X = G/P$, where $G = SL_{n+1}$ and $P = \text{stab}[1, 0, \dots, 0]$. Let $T = \{(t_1, \dots, t_{n+1}) \in (\mathbb{C}^*)^{n+1} \mid \prod t_i = 1\}$ be the maximal torus of G acting on X in the obvious way. Let B denote the set of upper triangular matrices in G . Let $e^{\varepsilon_i} \in \hat{T}$ be defined by $e^{\varepsilon_i}(t_1, \dots, t_{n+1}) = t_i$, where \hat{T} is the group of characters of T . Written additively, we denote e^{ε_i} by ε_i itself. Then, the set of positive roots is $\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\}$. Let $\chi_i = \varepsilon_1 + \dots + \varepsilon_i$, $i = 1, \dots, n+1$; for $i \leq n$ these are the fundamental (dominant) weights. The elements of W^P can be identified with the set of integers $[n] := \{0, 1, \dots, n\}$. In this section we deviate from our convention in the rest of the paper and, for any $u \in [n]$, simply write X_u , X^u , ξ^u for X_u^P , X_P^u and ξ_P^u respectively. Then, $X_u = \{[x_1, \dots, x_{u+1}, 0, \dots, 0]\}$ and $X^u = \{[0, \dots, 0, x_{u+1}, \dots, x_{n+1}]\}$. Note that $\dim X_u = \text{codim } X^u = u$, and the intersections $X_u \cap X^u$ are transverse. We will write simply $p_{u,v}^w$ for the structure constants with respect to the basis $\{\xi^u\}$ of $K_T(\mathbb{P}^n)$, and $b_{u,v}^w$ for the structure constants with respect to the basis $\{[\mathcal{O}_{X_u}]\}$.

Let $V(\mu_1, \dots, \mu_k)$ denote the T -module with weights μ_1, \dots, μ_k . Let $E_p(x_1, \dots, x_r)$ denote the p -th elementary symmetric function in the variables x_1, \dots, x_r . Recall that D denotes the diagonal embedding of X in $X \times X$.

Lemma 6.1. *If $\gamma \in K_T(X)$, then, for any $u, w \in [n]$,*

$$\begin{aligned} \chi(X \times X, D_*[\mathcal{O}_{X_w}] \otimes ([\mathcal{O}_{X^u}(-\partial X^u)] \boxtimes \gamma)) \\ = \chi(X_w \cap X^u, \gamma) - \chi(X_w \cap \partial X^u, \gamma). \end{aligned}$$

Proof. Arguing as in the proof of Proposition 3.6, we get that the left hand side equals $\chi(X, [\mathcal{O}_{X_w}] \otimes [\mathcal{O}_{X^u}(-\partial X^u)] \otimes \gamma)$. This in turn equals

$$\chi(X, [\mathcal{O}_{X_w}] \otimes [\mathcal{O}_{X^u}] \otimes \gamma) - \chi(X, [\mathcal{O}_{X_w}] \otimes [\mathcal{O}_{\partial X^u}] \otimes \gamma).$$

Since X_w intersects X^u and ∂X^u properly and these varieties are Cohen-Macaulay, Brion's result [Bri02, Lemma 1] implies that $[\mathcal{O}_{X_w}] \otimes [\mathcal{O}_{X^u}] = [\mathcal{O}_{X_w \cap X^u}]$ and $[\mathcal{O}_{X_w}] \otimes [\mathcal{O}_{\partial X^u}] = [\mathcal{O}_{X_w \cap \partial X^u}]$. Hence the above difference equals

$$\chi(X, [\mathcal{O}_{X_w \cap X^u}] \otimes \gamma) - \chi(X, [\mathcal{O}_{X_w \cap \partial X^u}] \otimes \gamma).$$

In general, if Y is a closed subscheme of X , the projection formula implies that $\chi(X, [\mathcal{O}_Y] \otimes \gamma) = \chi(Y, \gamma)$. The result follows. \square

For any $n \in \mathbb{Z}$, the character $e^{n\varepsilon_1}$ of T extends to a character of P . As earlier, let $\mathbb{C}_{n\varepsilon_1}$ denote the corresponding P -module and let $\mathcal{L}(n\varepsilon_1)$ denote the line bundle $G \times_P \mathbb{C}_{-n\varepsilon_1}$ on $X = G/P = \mathbb{P}^n$.

Lemma 6.2. *For any $n \in \mathbb{Z}$, $\mathcal{L}(n\varepsilon_1) \cong \mathcal{O}_X(n)$ as G -equivariant line bundles, where $\mathcal{O}_X(1)$ denotes the dual of the tautological bundle.*

Proof. Both $\mathcal{L}(n\varepsilon_1)$ and $\mathcal{O}_X(n)$ are sheaves of sections of G -equivariant line bundles. Hence, the line bundles are determined by the character of P on the fiber over the P -fixed point $[1, 0, \dots, 0]$. Since P acts by the character $e^{-n\varepsilon_1}$ on each line bundle, the line bundles are isomorphic. \square

Lemma 6.3. *If $w \geq u$, then as T -equivariant coherent sheaves on $X_w \cap X^u$,*

$$\omega_{X_w \cap X^u} = \mathcal{L}((-w + u - 1)\varepsilon_1)|_{X_w \cap X^u} \otimes e^{X^u - X^{w+1}}.$$

Proof. Observe first that $X_w \cap X^u = \{[0, \dots, 0, x_{u+1}, \dots, x_{w+1}, 0, \dots, 0]\}$. In particular, $X_w \cap X^u$ is a projective space. To prove the lemma, we will show more generally that if T acts on a vector space V with weights μ_1, \dots, μ_{k+1} , then, as T -equivariant sheaves on $\mathbb{P}(V)$, we have

$$\omega_{\mathbb{P}(V)} \cong e^{-(\mu_1 + \dots + \mu_{k+1})} \mathcal{O}_{\mathbb{P}(V)}(-(k+1)),$$

where $\mathbb{P}(V)$ denotes the space of lines in V . This is a consequence of [Har77, Ex. III.8.4]; a direct proof is as follows. We know that non-equivariantly $\omega_{\mathbb{P}(V)} \cong \mathcal{O}_{\mathbb{P}(V)}(-(k+1))$. So, equivariantly we must have

$\omega_{\mathbb{P}(V)} \cong e^\mu \otimes \mathcal{O}_{\mathbb{P}(V)}(-(k+1))$, for some character e^μ of T . To determine μ , observe that T acts on $H^k(\mathbb{P}(V), \omega_{\mathbb{P}(V)})$ by the trivial character $e^0 = 1$. This holds since, by Serre duality, $H^k(\mathbb{P}(V), \omega_{\mathbb{P}(V)}) \cong H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)})^*$. Moreover, this isomorphism is T -equivariant because the Serre duality is natural; since T acts trivially on $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)})$, the claim follows. On the other hand, we claim that T acts on $H^k(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(-(k+1)))$ with weight $\mu_1 + \cdots + \mu_{k+1}$. To prove this, observe that on the open set U_i where the i -th coordinate x_i is nonzero, we have a section

$$\tau_i : [x_1, \dots, x_{k+1}] \mapsto \left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \dots, \frac{x_{k+1}}{x_i} \right)$$

of $\mathcal{O}_{\mathbb{P}(V)}(-1)$, on which T acts with weight μ_i . A generator of $H^k(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(-(k+1)))$ is represented (as a Čech cocycle) by the section $\tau_1 \tau_2 \cdots \tau_{k+1} \in \Gamma(U_1 \cap \cdots \cap U_{k+1}, \mathcal{O}(-(k+1)))$. Since T acts on this section with weight $\mu_1 + \cdots + \mu_{k+1}$, the claim follows. We conclude that, T -equivariantly,

$$\mathcal{O}_{\mathbb{P}(V)}(-(k+1)) \cong \omega_{\mathbb{P}(V)} \otimes e^{\mu_1 + \cdots + \mu_{k+1}},$$

proving the lemma. \square

Let $Y_i \subset X$ be defined by the equation $x_i = 0$.

Lemma 6.4. $[\mathcal{O}_{Y_i}] = 1 - e^{-\varepsilon_i}[\mathcal{L}(-\varepsilon_1)]$ in $K_T(X)$.

Proof. By Lemma 6.2, $\mathcal{L}(-\varepsilon_1) = \mathcal{O}_X(-1)$. Let \mathcal{I}_{Y_i} denote the ideal sheaf of Y_i . Since $[\mathcal{O}_{Y_i}] = 1 - [\mathcal{I}_{Y_i}]$, it suffices to show that $\mathcal{I}_{Y_i} \cong e^{-\varepsilon_i} \mathcal{O}_X(-1)$ as T -equivariant coherent sheaves on X . On the open set $U_j : x_j \neq 0$, we have affine coordinates $\frac{x_k}{x_j}$ ($k \neq j$). Now, $\mathcal{I}_{Y_i}(U_j)$ is generated by the section $\sigma_j = \frac{x_i}{x_j}$, which transforms under T by the weight $\varepsilon_j - \varepsilon_i$. Similarly, $\mathcal{O}_X(-1)(U_j)$ is generated by the section

$$\tau_j : [x_1, \dots, x_{n+1}] \mapsto \left(\frac{x_1}{x_j}, \frac{x_2}{x_j}, \dots, \frac{x_{n+1}}{x_j} \right),$$

which transforms under T by the weight ε_j . Since

$$\frac{\sigma_j}{\tau_j} = \frac{\tau_k}{\tau_j} = \frac{x_j}{x_k},$$

it follows that the map $\mathcal{O}_X(-1) \otimes e^{-\varepsilon_i} \rightarrow \mathcal{I}_{Y_i}$ defined on U_j by $\tau_j \otimes 1 \mapsto \sigma_j$ is a T -equivariant sheaf isomorphism. \square

Remark 6.5. More generally, the following is true. Let Y be any T -scheme and let \mathcal{L} be a T -equivariant line bundle on Y . Given a section σ of weight λ and zero scheme $Z(\sigma)$, we have

$$[\mathcal{O}_{Z(\sigma)}] = 1 - e^\lambda[\mathcal{L}^*].$$

See Proposition 7.3 for a related result.

Corollary 6.6.

$$\xi^v = e^{-\varepsilon_{v+1}} [\mathcal{L}(-\varepsilon_1)] \prod_{i=1}^v (1 - e^{-\varepsilon_i} [\mathcal{L}(-\varepsilon_1)]) \quad (0 \leq v < n)$$

$$\xi^n = \prod_{i=1}^n (1 - e^{-\varepsilon_i} [\mathcal{L}(-\varepsilon_1)]).$$

Proof. $\xi^v = [\mathcal{O}_{X^v}(-\partial X^v)] = [\mathcal{O}_{X^v}] - [\mathcal{O}_{X^{v+1}}]$. For $1 \leq v \leq n$, since X^v is the transverse intersection of Y_1, \dots, Y_v , we have by Lemma 6.4,

$$[\mathcal{O}_{X^v}] = \prod_{i=1}^v [\mathcal{O}_{Y_i}] = \prod_{i=1}^v (1 - e^{-\varepsilon_i} [\mathcal{L}(-\varepsilon_1)]).$$

(For $v = 0$, this formula is interpreted as saying that $[\mathcal{O}_{X^0}] = 1$, which is true since $X^0 = X$.) A similar equation holds for $[\mathcal{O}_{X^{v+1}}]$ (with $[\mathcal{O}_{X^{n+1}}] = 0$, as X^{n+1} is empty). Subtracting the two formulae gives the result. \square

6.2. Structure constants with respect to the dual structure sheaf basis for $K_T(\mathbb{P}^n)$. Write $[\sum_i a_i t^i]_p = a_p$ and $[\sum_{i,j} b_{i,j} s^i t^j]_{p,q} = b_{p,q}$. We have the following explicit formula for the structure constants with respect to the dual structure sheaf basis.

Theorem 6.7. *For any $0 \leq u, v, w \leq n$,*

$$(-1)^{u+v+w} p_{u,v}^w$$

$$= e^{\chi_{w+1} - \chi_{u+1} - \chi_{v+1}} \left[\frac{(\prod_{i=1}^u (1 - te^{\varepsilon_i})) (\prod_{i=1}^v (1 - te^{\varepsilon_i}))}{\prod_{i=1}^{w+1} (1 - te^{\varepsilon_i})} \right]_{u+v-w+1}.$$

Proof. We have

$$(25) \quad p_{u,v}^w = \chi(X \times X, D_*[\mathcal{O}_{X_w}] \otimes ([\mathcal{O}_{X^u}(-\partial X^u)] \boxtimes \xi^v))$$

$$= \chi(X_w \cap X^u, \xi^v) - \chi(X_w \cap X^{u+1}, \xi^v),$$

where the first equality is by Propositions 3.6 and 2.1 and the second is by Lemma 6.1. Suppose first that $v < n$. (We separate the case $v = n$ because the formula for ξ^v is different in this case.) We have, by

Corollary 6.6,

$$\begin{aligned}
\chi(X_w \cap X^u, \xi^v) &= e^{-\varepsilon_{v+1}} \chi(X_w \cap X^u, [\mathcal{L}(-\varepsilon_1)] \prod_{i=1}^v (1 - e^{-\varepsilon_i} [\mathcal{L}(-\varepsilon_1)])) \\
&= e^{-\varepsilon_{v+1}} \chi\left(X_w \cap X^u, \sum_{p=0}^v (-1)^p E_p(e^{-\varepsilon_1}, \dots, e^{-\varepsilon_v}) [\mathcal{L}(-(p+1)\varepsilon_1)]\right) \\
&= \sum_{p=0}^v (-1)^p e^{-\varepsilon_{v+1}} E_p(e^{-\varepsilon_1}, \dots, e^{-\varepsilon_v}) \chi(X_w \cap X^u, [\mathcal{L}(-(p+1)\varepsilon_1)]).
\end{aligned}$$

We apply Serre duality to this formula, using the formula of Lemma 6.3 for the dualizing sheaf. Thus, the above sum can be expressed as (note that $X_w \cap X^u$ is nonempty iff $w \geq u$ and $\dim X_w \cap X^u = w - u$):

$$(-1)^{w-u} \sum_{p=0}^v (-1)^p e^{-\varepsilon_{v+1}} E_p(e^{-\varepsilon_1}, \dots, e^{-\varepsilon_v}) \bar{\chi}(X_w \cap X^u, [\mathcal{L}((p-w+u)\varepsilon_1)] \otimes e^{\chi_u - \chi_{w+1}}).$$

In this expression, the only contribution comes from the cohomology in degree 0 (this follows from [Har77, Theorem III.5.1], since $X_w \cap X^u$ is a projective space). So, the above expression reduces to

$$\begin{aligned}
&(-1)^{w+u} e^{\chi_{w+1} - \chi_u - \varepsilon_{v+1}} \sum_{p=0}^v (-1)^p E_p(e^{-\varepsilon_1}, \dots, e^{-\varepsilon_v}) \\
&\quad \times \bar{h}^0(X_w \cap X^u, \mathcal{L}((p-w+u)\varepsilon_1)) \\
&= (-1)^{w+u} e^{\chi_{w+1} - \chi_u - \varepsilon_{v+1}} \sum_{p=0}^v (-1)^p E_p(e^{-\varepsilon_1}, \dots, e^{-\varepsilon_v}) \\
&\quad \times \text{ch}\left(S^{p-w+u}(V(\varepsilon_{u+1}, \dots, \varepsilon_{w+1}))\right).
\end{aligned}$$

Form the generating function

$$\begin{aligned}
f(s, t) &= \sum_{p,q} (-1)^p E_p(e^{-\varepsilon_1}, \dots, e^{-\varepsilon_v}) \text{ch}\left(S^q(V(\varepsilon_{u+1}, \dots, \varepsilon_{w+1}))\right) s^p t^q \\
&= \left(\prod_{i=1}^v (1 - s e^{-\varepsilon_i})\right) \left(\prod_{j=u+1}^{w+1} (1 - t e^{\varepsilon_j})^{-1}\right),
\end{aligned}$$

where the expansion of $(1 - t e^{\varepsilon_j})^{-1}$ is at $t = 0$ (i.e., as a power series in t). Then,

$$\chi(X_w \cap X^u, \xi^v) = (-1)^{w+u} e^{\chi_{w+1} - \chi_u - \varepsilon_{v+1}} \sum_p [f(s, t)]_{p, p-w+u}.$$

Setting $s = t^{-1}$, we obtain

$$\begin{aligned}
\chi(X_w \cap X^u, \xi^v) &= (-1)^{w+u} e^{\chi_{w+1} - \chi_u - \varepsilon_{v+1}} [f(t^{-1}, t)]_{u-w} \\
&= (-1)^{w+u} e^{\chi_{w+1} - \chi_u - \varepsilon_{v+1}} \left[t^v \left(\prod_{i=1}^v (1 - t^{-1} e^{-\varepsilon_i}) \right) \left(\prod_{j=u+1}^{w+1} (1 - t e^{\varepsilon_j})^{-1} \right) \right]_{u+v-w} \\
&= (-1)^{u+v+w} e^{\chi_{w+1} - \chi_u - \varepsilon_{v+1}} \left[\left(\prod_{i=1}^v (e^{-\varepsilon_i} - t) \right) \left(\prod_{j=u+1}^{w+1} (1 - t e^{\varepsilon_j})^{-1} \right) \right]_{u+v-w} \\
&= (-1)^{u+v+w} e^{\chi_{w+1} - \chi_u - \chi_{v+1}} \left[\left(\prod_{i=1}^v (1 - t e^{\varepsilon_i}) \right) \left(\prod_{j=u+1}^{w+1} (1 - t e^{\varepsilon_j})^{-1} \right) \right]_{u+v-w},
\end{aligned}$$

where multiplying by t^v enabled us to shift degree from $u-w$ to $u+v-w$, and in the last step we have used the equation $\chi_{v+1} = \varepsilon_1 + \dots + \varepsilon_{v+1}$. From this last expression, we obtain

$$\begin{aligned}
&\chi(X_w \cap X^u, \xi^v) \\
&= (-1)^{u+v+w} e^{\chi_{w+1} - \chi_u - \chi_{v+1}} \left[\frac{(\prod_{i=1}^u (1 - t e^{\varepsilon_i})) (\prod_{i=1}^v (1 - t e^{\varepsilon_i}))}{\prod_{i=1}^{w+1} (1 - t e^{\varepsilon_i})} \right]_{u+v-w}.
\end{aligned}$$

Set

$$A = \frac{(\prod_{i=1}^u (1 - t e^{\varepsilon_i})) (\prod_{i=1}^v (1 - t e^{\varepsilon_i}))}{\prod_{i=1}^{w+1} (1 - t e^{\varepsilon_i})}.$$

Replacing u by $u+1$ in the formula for $\chi(X_w \cap X^u, \xi^v)$ yields an expression for $\chi(X_w \cap X^{u+1}, \xi^v)$ (the expression turns out to be 0 if $u = n$, so we can allow the case $u = n$ in the argument). The difference is

$$\begin{aligned}
&\chi(X_w \cap X^u, \xi^v) - \chi(X_w \cap X^{u+1}, \xi^v) \\
&= (-1)^{u+v+w} e^{\chi_{w+1} - \chi_u - \chi_{v+1}} [A]_{u+v-w} \\
&\quad - (-1)^{u+v+w+1} e^{\chi_{w+1} - \chi_{u+1} - \chi_{v+1}} [A(1 - t e^{\varepsilon_{u+1}})]_{u+v-w+1} \\
&= (-1)^{u+v+w} e^{\chi_{w+1} - \chi_u - \chi_{v+1}} \left([tA]_{u+v-w+1} + e^{-\varepsilon_{u+1}} [A(1 - t e^{\varepsilon_{u+1}})]_{u+v-w+1} \right) \\
&= (-1)^{u+v+w} e^{\chi_{w+1} - \chi_u - \chi_{v+1}} [A(t + e^{-\varepsilon_{u+1}}(1 - t e^{\varepsilon_{u+1}}))]_{u+v-w+1} \\
(26) \quad &= (-1)^{u+v+w} e^{\chi_{w+1} - \chi_u - \varepsilon_{u+1} - \chi_{v+1}} [A]_{u+v-w+1}.
\end{aligned}$$

Since $\chi_u + \varepsilon_{u+1} = \chi_{u+1}$, we obtain the desired expression from (25). This proves the result if $v < n$. Interchanging the roles of u and v proves the result if $u < n$. The only remaining case is $u = v = n$. Rather than imitate the argument above for this case, we will simply calculate both sides of the equation in the statement of the theorem.

Since $\xi^n = [\mathcal{O}_{X^n}]$ and $X^n = \{[0, \dots, 0, 1]\}$, the self-intersection formula (cf. [VV06, Theorem 2.1]) implies that

$$\xi^n \xi^n = \left(\prod_{i=1}^n (1 - e^{\varepsilon_{n+1} - \varepsilon_i}) \right) \xi^n.$$

So

$$(27) \quad p_{n,n}^w = 0 \quad \text{if } w < n \quad \text{and}$$

$$(28) \quad (-1)^n p_{n,n}^n = (-1)^n \prod_{i=1}^n (1 - e^{\varepsilon_{n+1} - \varepsilon_i}) = \prod_{i=1}^n (e^{\varepsilon_{n+1} - \varepsilon_i} - 1).$$

Observe that the right side of the equation in the statement of the theorem is 0 unless $w = n$ (see the proof of Corollary 6.8 below). So, it suffices to calculate this side if $w = n$. The case $n = 1$ gives the correct result (we omit the calculation). Assume the result holds for $n - 1$. We have

$$\begin{aligned} & e^{-\chi_{n+1}} \left[\frac{\prod_{i=1}^n (1 - te^{\varepsilon_i})}{1 - te^{\varepsilon_{n+1}}} \right]_{n+1} \\ &= e^{-\chi_{n+1}} \left[\frac{\prod_{i=1}^n (1 - te^{\varepsilon_i})}{1 - te^{\varepsilon_{n+1}}} - \frac{(1 - te^{\varepsilon_{n+1}}) \cdot \prod_{i=2}^n (1 - te^{\varepsilon_i})}{1 - te^{\varepsilon_{n+1}}} \right. \\ & \quad \left. + \frac{(1 - te^{\varepsilon_{n+1}}) \cdot \prod_{i=2}^n (1 - te^{\varepsilon_i})}{1 - te^{\varepsilon_{n+1}}} \right]_{n+1} \\ &= e^{-\chi_{n+1}} \left[\frac{t(e^{\varepsilon_{n+1}} - e^{\varepsilon_1}) \cdot \prod_{i=2}^n (1 - te^{\varepsilon_i})}{1 - te^{\varepsilon_{n+1}}} + \prod_{i=2}^n (1 - te^{\varepsilon_i}) \right]_{n+1}. \end{aligned}$$

The second product is of degree $n - 1$ in t and therefore contributes nothing to the degree $n + 1$ part. So, the above expression equals

$$\begin{aligned} & e^{-\chi_{n+1}} (e^{\varepsilon_{n+1}} - e^{\varepsilon_1}) \left[\frac{\prod_{i=2}^n (1 - te^{\varepsilon_i})}{1 - te^{\varepsilon_{n+1}}} \right]_n \\ &= (e^{\varepsilon_{n+1} - \varepsilon_1} - 1) e^{-\chi_{n+1} + \varepsilon_1} \left[\frac{\prod_{i=2}^n (1 - te^{\varepsilon_i})}{1 - te^{\varepsilon_{n+1}}} \right]_n \\ &= (e^{\varepsilon_{n+1} - \varepsilon_1} - 1) \prod_{i=2}^n (e^{\varepsilon_{n+1} - \varepsilon_i} - 1), \end{aligned}$$

where in the last step we have used our inductive hypothesis. The result follows from identity (28). \square

Corollary 6.8. *If $p_{u,v}^w \neq 0$, then $u, v \leq w \leq u + v + 1$.*

Proof. Keeping the notation of the previous proof, we observe that the power series expansion of A contains no negative powers of t . Hence, if $p_{u,v}^w \neq 0$ then, by the previous theorem, $u + v - w + 1 \geq 0$, i.e., $w \leq u + v + 1$. Next, if $w \leq u - 1$, then A is a polynomial of degree $u + v - w - 1$. Hence, $[A]_{u+v-w+1} = 0$, so $p_{u,v}^w = 0$. Thus, if $p_{u,v}^w \neq 0$ then $u \leq w$; similarly, $v \leq w$. \square

Corollary 6.9.

$$\begin{aligned} p_{u,0}^w &= 0 \quad \text{if} \quad u \neq w - 1, w, \\ p_{w-1,0}^w &= -e^{\varepsilon_{w+1} - \varepsilon_1}, \quad \text{for } w \geq 1 \\ p_{w,0}^w &= e^{\varepsilon_{w+1} - \varepsilon_1}. \end{aligned}$$

Proof. The first statement is immediate from Corollary 6.8. The other formulas are easy consequences of Theorem 6.7. \square

Corollary 6.10. *If $n, m \geq w$ then the structure constants $p_{u,v}^w$ are the same for \mathbb{P}^n and \mathbb{P}^m .*

Proof. This follows immediately from the expression for $p_{u,v}^w$ given by the theorem. \square

If $w \leq n$ and μ_1, \dots, μ_{n+1} are weights of T , let $p_{u,v}^w(\mu_1, \dots, \mu_{n+1})$ denote the element of $R(T)$ obtained by replacing ε_i by μ_i in the expression of the theorem for $p_{u,v}^w$, for $i = 1, \dots, n + 1$.

Define $\tilde{p}_{u,v}^w = (-1)^{u+v+w} p_{u,v}^w$. The next theorem gives a recurrence for the $\tilde{p}_{u,v}^w$. We set $\tilde{p}_{u,v}^w = 0$ if u, v or w is negative.

Theorem 6.11. *If $v \geq 1$, then*

$$\tilde{p}_{u,v}^w = (e^{\varepsilon_{u+1} - \varepsilon_1} - 1) \tilde{p}_{u-1,v-1}^{w-1}(\varepsilon_2, \dots, \varepsilon_{n+1}) + e^{\varepsilon_{u+2} - \varepsilon_1} \tilde{p}_{u,v-1}^{w-1}(\varepsilon_2, \dots, \varepsilon_{n+1}).$$

Proof. Let A be as in the proof of Theorem 6.7. We can cancel common factors from the numerator and denominator to write

$$A = \frac{\prod_{i=1}^v (1 - te^{\varepsilon_i})}{\prod_{j=u+1}^{w+1} (1 - te^{\varepsilon_j})}.$$

Let

$$B = \frac{(1 - te^{\varepsilon_{u+1}}) \prod_{i=2}^v (1 - te^{\varepsilon_i})}{\prod_{j=u+1}^{w+1} (1 - te^{\varepsilon_j})}.$$

By Theorem 6.7,

$$\begin{aligned} \tilde{p}_{u,v}^w &= e^{\chi_{w+1} - \chi_{u+1} - \chi_{v+1}} [A]_{u+v+1-w} \\ &= e^{\chi_{w+1} - \chi_{u+1} - \chi_{v+1}} [A - B + B]_{u+v+1-w}. \end{aligned}$$

We have

$$\begin{aligned} & e^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}}[A-B]_{u+v+1-w} \\ &= e^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}} \left[\frac{t(e^{\varepsilon_{u+1}} - e^{\varepsilon_1}) \prod_{i=2}^v (1 - te^{\varepsilon_i})}{\prod_{j=u+1}^{w+1} (1 - te^{\varepsilon_j})} \right]_{u+v+1-w}. \end{aligned}$$

We can get rid of the factor t in the numerator by taking the part in degree $u + v - w$. Since $e^{\varepsilon_{u+1}} - e^{\varepsilon_1} = e^{\varepsilon_1}(e^{\varepsilon_{u+1}-\varepsilon_1} - 1)$, we see that the above expression equals

$$(e^{\varepsilon_{u+1}-\varepsilon_1} - 1) e^{\chi_{w+1}+\varepsilon_1-\chi_{u+1}-\chi_{v+1}} \left[\frac{\prod_{i=2}^v (1 - te^{\varepsilon_i})}{\prod_{j=u+1}^{w+1} (1 - te^{\varepsilon_j})} \right]_{u+v-w}.$$

Using the definition $\chi_k = \varepsilon_1 + \dots + \varepsilon_k$, we see that the above expression equals

$$(e^{\varepsilon_{u+1}-\varepsilon_1} - 1) \tilde{p}_{u-1,v-1}^{w-1}(\varepsilon_2, \dots, \varepsilon_{n+1}).$$

Next, we have

$$\begin{aligned} (29) \quad & e^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}}[B]_{u+v+1-w} \\ &= e^{\chi_{w+1}-\chi_{u+1}-\chi_{v+1}} \left[\frac{\prod_{i=2}^v (1 - te^{\varepsilon_i})}{\prod_{j=u+2}^{w+1} (1 - te^{\varepsilon_j})} \right]_{u+v+1-w}. \end{aligned}$$

Now,

$$\begin{aligned} \chi_{w+1} - \chi_{u+1} - \chi_{v+1} &= (\varepsilon_1 + \dots + \varepsilon_{w+1}) - (\varepsilon_1 + \dots + \varepsilon_{u+1}) \\ &\quad - (\varepsilon_1 + \dots + \varepsilon_{v+1}) \\ &= (\varepsilon_2 + \dots + \varepsilon_{w+1}) - (\varepsilon_2 + \dots + \varepsilon_{u+2}) \\ &\quad - (\varepsilon_2 + \dots + \varepsilon_{v+1}) + (\varepsilon_{u+2} - \varepsilon_1). \end{aligned}$$

Using this we see that the expression (29) equals

$$e^{\varepsilon_{u+2}-\varepsilon_1} \tilde{p}_{u,v-1}^{w-1}(\varepsilon_2, \dots, \varepsilon_{n+1}).$$

This proves the theorem. \square

As an immediate consequence of these results, we can verify Conjecture 3.1 for projective space:

Theorem 6.12. *For all $0 \leq u, v, w \leq n$,*

$$\tilde{p}_{u,v}^w \in \mathbb{Z}_+[e^{-\alpha} - 1]_{\alpha \in \Delta^+}.$$

Proof. This holds if $v = 0$ by Corollary 6.9. The general case follows by induction using the recurrence of Theorem 6.11. \square

Remark 6.13. (a) For $0 \leq u, v, w \leq n$, define

$$\tilde{q}_{u,v}^w = (-1)^{u+v+w} \chi(X_w \cap X^u, \xi^v).$$

Then, by the same proof as that of Theorems 6.7 and 6.11, we have the recursion

$$\tilde{q}_{u,v}^w = (e^{\varepsilon_{u+1}-\varepsilon_1} - 1) \tilde{q}_{u-1,v-1}^{w-1}(\varepsilon_2, \dots, \varepsilon_{n+1}) + e^{\varepsilon_{u+1}-\varepsilon_1} \tilde{q}_{u,v-1}^{w-1}(\varepsilon_2, \dots, \varepsilon_{n+1}).$$

So, by induction on w , we get that

$$\tilde{q}_{u,v}^w \in \mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}.$$

(b) As a consequence of Theorem 6.11 and Corollary 6.9, we get that in the non-equivariant K -theory $K(\mathbb{P}^n)$, we have

$$p_{u,v}^w = 0, \text{ for } u + v > w$$

and

$$p_{u,w-u}^w = 1, \text{ for any } u \leq w; \quad p_{u,w-u-1}^w = -1, \text{ for any } u \leq w - 1.$$

6.3. Structure constants with respect to the structure sheaf basis of $K_T(\mathbb{P}^n)$. We give explicit formulas for the structure constants with respect to the structure sheaf basis. These are strikingly similar to the formulas for the structure constants in the dual structure sheaf basis, but they differ subtly. We will state these formulas here. We omit most details of the proofs, which are very similar to the proofs in the previous subsection.

Let w_o denote the longest element of the Weyl group of SL_{n+1} , so $w_o(\varepsilon_i) = \varepsilon_{n+2-i}$. For $u \in [n]$, let $\bar{u} = n - u$. To state our formulas, it will be convenient to introduce the notation $r_{u,v}^w = w_o(b_{\bar{u},\bar{v}}^{\bar{w}})$.

Theorem 6.14. For any $0 \leq u, v, w \leq n$,

$$\begin{aligned} & (-1)^{u+v+w} r_{u,v}^w \\ &= e^{\chi_w - \chi_u - \chi_v} \left[\frac{(\prod_{i=1}^u (1 - te^{\varepsilon_i})) (\prod_{i=1}^v (1 - te^{\varepsilon_i}))}{\prod_{i=1}^{w+1} (1 - te^{\varepsilon_i})} \right]_{u+v-w}. \end{aligned}$$

Proof. Using Remark 3.11 and Proposition 2.1, we see that

$$b_{\bar{u},\bar{v}}^{\bar{w}} = \chi(X \times X, D_* \xi^{\bar{w}} \otimes ([\mathcal{O}_{X_{\bar{u}}}] \boxtimes [\mathcal{O}_{X_{\bar{v}}}])).$$

Arguing as in the proof of Proposition 2.1, we see that the right side of the above identity is equal to

$$\chi(X \times X, D_* [\mathcal{O}_{X_{\bar{u}}}] \otimes (\xi^{\bar{w}} \boxtimes [\mathcal{O}_{X_{\bar{v}}}])).$$

The case $w = 0$ of the theorem can be checked separately, so assume $w > 0$, i.e., $\bar{w} < n$. Then, the above expression equals

$$\chi(X_{\bar{u}} \cap X^{\bar{w}}, [\mathcal{O}_{X_{\bar{v}}}]) - \chi(X_{\bar{u}} \cap X^{\bar{w}+1}, [\mathcal{O}_{X_{\bar{v}}}]).$$

This can be calculated as in the proof of Theorem 6.7; we omit the details. \square

Arguing as in the proof of Corollary 6.8 gives the following result.

Corollary 6.15. *If $r_{u,v}^w \neq 0$, then $u, v \leq w \leq u + v$.*

The next result gives “initial condition” for the $r_{u,v}^w$.

Proposition 6.16. $r_{u,0}^w = \delta_{u,w}$.

Proof. This follows because $[\mathcal{O}_{X_n}]$ is the identity element in $K_T(\mathbb{P}^n)$. (Alternatively, the proposition can be deduced from Theorem 6.14.) \square

Write $\tilde{r}_{u,v}^w = (-1)^{u+v+w} r_{u,v}^w$.

Theorem 6.17. *If $v \geq 1$, then*

$$\tilde{r}_{u,v}^w = (e^{\varepsilon_{u+1}-\varepsilon_1} - 1) \tilde{r}_{u-1,v-1}^{w-1}(\varepsilon_2, \dots, \varepsilon_{n+1}) + e^{\varepsilon_{u+1}-\varepsilon_1} \tilde{r}_{u,v-1}^{w-1}(\varepsilon_2, \dots, \varepsilon_{n+1}).$$

Proof. This is similar to the proof of Theorem 6.11; we omit the details. \square

The above two results imply that Conjecture 3.10 holds for projective spaces (cf. Remark 3.11).

Theorem 6.18. *For all $0 \leq u, v, w \leq n$,*

$$\tilde{r}_{u,v}^w \in \mathbb{Z}_+[e^{-\alpha} - 1]_{\alpha \in \Delta^+}.$$

Hence,

$$(-1)^{n+u+v+w} b_{u,v}^w \in \mathbb{Z}_+[e^{\alpha} - 1]_{\alpha \in \Delta^+}.$$

Proof. The first result follows from Proposition 6.16 and Theorem 6.17. The second result follows from the first, since w_o takes negative roots to positive roots. \square

7. A MORE GENERAL POSITIVITY CONJECTURE

We revert to the notation and assumptions of Section 3. The following conjecture is an equivariant generalization of [Bri02, Theorem 1]. By Proposition 3.6, this conjecture, with $G \times G$ in place of G and T' equal to the diagonal torus in $T \times T$, would imply Conjecture 3.1.

Conjecture 7.1. Let T' be a subtorus of T and let $Y \subset G/P$ be a T' -stable irreducible subvariety with rational singularities. Express, in $K_{T'}(G/P)$,

$$[\mathcal{O}_Y] = \sum_{w \in W^P} a_w^Y [\mathcal{O}_{X_w^P}].$$

Then,

$$(-1)^{\text{codim } Y + \text{codim } X_w^P} a_w^Y \in \mathbb{Z}_+[e_{|T'}^{-\beta} - 1]_{\beta \in \Delta^+}.$$

Remark 7.2. 1) By (a subsequent) Proposition 7.6 and Remark 7.7(a), the above conjecture is true for $Y = X_P^v \subset G/P$.

2) We have verified the above conjecture by an explicit calculation for $G = SL_3$, $P = B$ and $Y = X_w \cap X^v$ for any $v, w \in W$.

In the next proposition we view \mathbb{P}^1 as having projective coordinates $[x_0 : x_1]$, so $\frac{x_0}{x_1}$ is a rational function on \mathbb{P}^1 . We write $0 = [0 : 1]$ and $\infty = [1 : 0]$.

Proposition 7.3. *Suppose T acts on \mathbb{P}^1 such that 0 and ∞ are T -fixed and $\frac{x_0}{x_1}$ is a T -weight vector with weight $-\alpha$. Let X be an irreducible T -variety and $\phi : X \rightarrow \mathbb{P}^1$ a dominant T -equivariant morphism. Then, in $K_T(X)$,*

$$[\mathcal{O}_{\phi^{-1}(\infty)}] = (1 - e^\alpha)[\mathcal{O}_X] + e^\alpha[\mathcal{O}_{\phi^{-1}(0)}].$$

Proof. Since ϕ is a flat morphism (see [Har77, Ch. III, Prop. 9.7]), $\phi^*[\mathcal{O}_{\{0\}}] = [\mathcal{O}_{\phi^{-1}(0)}]$ and similarly for $\mathcal{O}_{\{\infty\}}$. Hence, it suffices to show that on \mathbb{P}^1 ,

$$[\mathcal{O}_{\{\infty\}}] = (1 - e^\alpha)[\mathcal{O}_{\mathbb{P}^1}] + e^\alpha[\mathcal{O}_{\{0\}}],$$

since applying ϕ^* gives the desired equation. We have exact sequences

$$0 \rightarrow \mathcal{I}_0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\{0\}} \rightarrow 0,$$

and

$$0 \rightarrow \mathcal{I}_\infty \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\{\infty\}} \rightarrow 0,$$

where \mathcal{I}_0 and \mathcal{I}_∞ are the ideal sheaves of $\{0\}$ and $\{\infty\}$, respectively. Nonequivariantly, $\mathcal{I}_0 = \mathcal{I}_\infty = \mathcal{O}_{\mathbb{P}^1}(-1)$, so $\mathcal{I}_0 \otimes \mathcal{I}_\infty^*$ is non-equivariantly isomorphic to $\mathcal{O}_{\mathbb{P}^1}$. Near $0 = [0 : 1]$ the sheaf \mathcal{I}_0 is generated by x_0/x_1 , which has weight $-\alpha$, and \mathcal{I}_∞ near 0 is generated by 1 . Hence, as T -equivariant sheaves, $\mathcal{I}_0 \simeq e^{-\alpha}\mathcal{I}_\infty$. So,

$$\begin{aligned} [\mathcal{O}_{\{\infty\}}] &= [\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{I}_{\{\infty\}}] = [\mathcal{O}_{\mathbb{P}^1}] - e^\alpha[\mathcal{I}_0] = [\mathcal{O}_{\mathbb{P}^1}] - e^\alpha([\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}_{\{0\}}]) \\ &= (1 - e^\alpha)[\mathcal{O}_{\mathbb{P}^1}] + e^\alpha[\mathcal{O}_{\{0\}}] \end{aligned}$$

as desired. \square

Let $w \in W$. If s is a simple reflection with $sw < w$, then $sX_w = X_w$ and hence $[\mathcal{O}_{sX_w}] = [\mathcal{O}_{X_w}]$. On the other hand, if $sw > w$ then we have the following result.

Proposition 7.4. *If s is a simple reflection with $sw > w$, then*

$$[\mathcal{O}_{sX_w}] = e^{-\alpha}[\mathcal{O}_{X_w}] - (e^{-\alpha} - 1)[\mathcal{O}_{X_{sw}}],$$

where α is the simple root corresponding to s .

Proof. Let P_s be the minimal parabolic corresponding to s . Consider

$$\begin{array}{ccc} P_s \times^B X_w & \xrightarrow{\mu} & X_{sw} \\ \downarrow \pi & & \\ P_s/B & = & \mathbb{P}^1, \end{array}$$

where μ takes the B -orbit $[p, x] \mapsto px$ and π takes $[p, x] \mapsto p \bmod B$. Then,

$$\pi^{-1}(0) = \{1\} \times X_w, \quad \pi^{-1}(\infty) = \{s\} \times X_w.$$

So, by Proposition 7.3,

$$[\mathcal{O}_{\{s\} \times X_w}] = e^{-\alpha}[\mathcal{O}_{\{1\} \times X_w}] + (1 - e^{-\alpha})[\mathcal{O}_{P_s \times^B X_w}].$$

Push forward the above identity to X_{sw} via μ to get the result. (Here we have used [BK05, Proposition 3.2.1].) \square

Lemma 7.5. *For any T -stable closed subscheme $Y \subset G/P$, write in $K_T(G/P)$,*

$$(30) \quad [\mathcal{O}_Y] = \sum_{w \in W^P} P_w [\mathcal{O}_{X_w^P}], \text{ for some (unique) } P_w \in R(T).$$

Then, for any $v \in W$,

$$[\mathcal{O}_{v^{-1}Y}] = \sum_{w \in W^P} (v^{-1}P_w) [\mathcal{O}_{v^{-1}X_w^P}].$$

Proof. Let $f : T \rightarrow T'$ be any homomorphism. If X is any scheme with T' -action, and T acts on X via f , then there is a map $f^* : K_{T'}(X) \rightarrow K_T(X)$ extending the natural pull-back map $R(T') \rightarrow R(T)$. For any T' -stable closed subscheme Y of X , f^* takes the class of \mathcal{O}_Y in $K_{T'}(X)$ to the class of \mathcal{O}_Y in $K_T(X)$. We now apply this to $X = G/P$ and $f : T \rightarrow T$ given by $f(t) = tvv^{-1}$. Since $f^*r = v^{-1}r$ for $r \in R(T)$, we get from (30) the equation

$$(31) \quad [\mathcal{O}_Y] = \sum_{w \in W^P} (v^{-1}P_w) [\mathcal{O}_{X_w^P}],$$

where in this equation T is viewed as acting on G/P through f . Write $(G/P, \odot)$ to indicate G/P with this new action of T .

Consider the automorphism

$$\phi_v : G/P \rightarrow (G/P, \odot), \quad gP \mapsto \dot{v}gP,$$

where \dot{v} is a representative of v in $N(T)$. This is T -equivariant, where T acts on the source G/P by the standard action. Then, $\phi_v^*[\mathcal{O}_{X_w^P}] = [\mathcal{O}_{v^{-1}X_w^P}]$ and $\phi_v^*[\mathcal{O}_Y] = [\mathcal{O}_{v^{-1}Y}]$. Since ϕ_v^* is $R(T)$ -linear, applying ϕ_v^* to (31) proves the result. \square

Proposition 7.6. *Write $[\mathcal{O}_{X^w}] = \sum_u e_{w,u} [\mathcal{O}_{X_u}]$. Then,*

$$(-1)^{\text{codim } X^w + \text{codim } X_u} e_{w,u} \in \mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}.$$

Proof. For any $v, w \in W$, write

$$[\mathcal{O}_{vX_w}] = \sum f_{w,u}^v [\mathcal{O}_{X_u}].$$

We prove by induction on $\ell(v)$, that for any $u, w \in W$,

$$(32) \quad (-1)^{\text{codim } X_w + \text{codim } X_u} f_{w,u}^v \in \mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}.$$

Of course, (32) is true for $v = e$. Now take vs_i with $\ell(vs_i) > \ell(v)$. If $s_i w < w$, then $[\mathcal{O}_{vX_w}] = [\mathcal{O}_{vs_i X_w}]$ and we are done. If $s_i w > w$, then, by Proposition 7.4,

$$[\mathcal{O}_{s_i X_w}] = e^{-\alpha_i} [\mathcal{O}_{X_w}] - (e^{-\alpha_i} - 1) [\mathcal{O}_{X_{s_i w}}].$$

Thus, by Lemma 7.5,

$$[\mathcal{O}_{vs_i X_w}] = e^{-v\alpha_i} [\mathcal{O}_{vX_w}] - (e^{-v\alpha_i} - 1) [\mathcal{O}_{vX_{s_i w}}].$$

Since $vs_i > v$, $v\alpha_i \in \Delta^+$. Moreover, by induction, for any $u \in W$, $(-1)^{\text{codim } X_w + \text{codim } X_u} f_{w,u}^v$ and $(-1)^{\text{codim } X_w - 1 + \text{codim } X_u} f_{s_i w, u}^v$ are in $\mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}$. Hence, $(-1)^{\text{codim } X_w + \text{codim } X_u} f_{w,u}^{vs_i} \in \mathbb{Z}_+[e^{-\beta} - 1]_{\beta \in \Delta^+}$. This completes the induction and hence (32) is proved for any $u, v, w \in W$. Since $X^w = w_o X_{w_o w}$, the proposition follows. \square

Remark 7.7. (a) For any standard parabolic P and any closed T -stable subvariety $Z \subset G/P$, since $\pi^*[\mathcal{O}_Z] = [\mathcal{O}_{\pi^{-1}(Z)}]$ (cf. the proof of Proposition 3.12), where $\pi : G/B \rightarrow G/P$ is the standard projection, the above proposition and (32) remain true for the Schubert varieties in G/P .

(b) Since any T -stable closed irreducible subvariety of \mathbb{P}^n (under the standard action of the maximal torus T of $SL(n+1)$) is a W -translate of a Schubert variety of \mathbb{P}^n , Conjecture 7.1 is true for any T -stable closed irreducible subvariety of \mathbb{P}^n (by virtue of (32)).

REFERENCES

- [BK05] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, vol. 231, Birkhäuser, Boston (2005).
- [BL03] M. Brion and V. Lakshmibai, *A geometric approach to standard monomial theory*, Represent. Theory **7** (2003), 651–680.
- [Bri02] M. Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), 137–159.
- [Buc02] A. Buch, *A Littlewood-Richardson rule for the K -theory of Grassmannians*, Acta Math. **189** (2002), 37–78.

- [CG97] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhuser, Boston, 1997.
- [Deo77] V. V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math. **39** (1977), 187–198.
- [Gra01] W. Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), 599–614.
- [Gra02] ———, *Equivariant k -theory and Schubert varieties*, preprint (2002).
- [Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag (1977).
- [KK90] B. Kostant and S. Kumar, *T -equivariant K -theory of generalized flag varieties*, J. Differential Geom. **32** (1990), 549–603.
- [KN98] S. Kumar and M. V. Nori, *Positivity of the cup product in cohomology of flag varieties associated to Kac-Moody groups*, Internat. Math. Res. Notices (1998), 757–763.
- [Ram87] A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, Inst. Hautes Études Sci. Publ. Math. **65** (1987), 61–90.
- [VV06] A. Vistoli and G. Vezzosi, *Higher algebraic K -theory of group actions with finite stabilizers*, Duke Math. J. **113** (2002), 1–55.
- [Wil06] M. Willems, *K -théorie équivariante des tours de Bott. Application à la structure multiplicative de la K -théorie équivariante des variétés de drapeaux*, Duke Math. J. **132** (2006), 271–309.

Addresses:

W.G.: Department of Mathematics, University of Georgia, Athens, GA 30602-7403, USA

S.K.: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA